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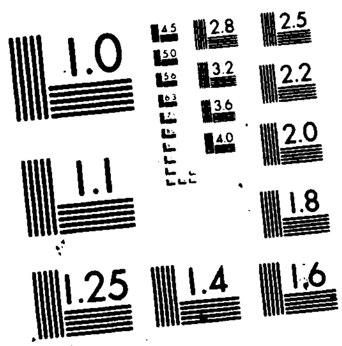

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BIFURCATION AND NONLINEAR OSCILLATIONS

by

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Two problems are discussed. The first one deals with the characterization of the flow for a periodic planar system which is the perturbation of an autonomous system which possesses either a saddle-node or degenerate focus or degenerate periodic orbit or homoclinic orbit. The second problem concerns the characterization of the flow near an equilibrium point of an autonomous equation when the linear variational equation has either two purely imaginary and one zero eigenvalue or two pairs of purely imaginary eigenvalues.

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S.N. Chow and J.K. Hale

ABSTRACT

Two problems are discussed. The first one deals with the characterization of the flow for a periodic planar system which is the perturbation of an autonomous system which possesses either a saddle-node or degenerate focus or degenerate periodic orbit or homoclinic orbit. The second problem concerns the characterization of the flow near an equilibrium point of an autonomous equation when the linear variational equation has either two purely imaginary and one zero eigenvalue or two pairs of purely imaginary eigenvalues.

0. Introduction

In recent years, there has been considerable interest in determining the manner in which complicated oscillatory phenomena can occur in dynamical systems through bifurcations (see, for example, [5],[6],[9],[10],[11],[12],[18]). In this paper, we consider the same type of problems for low dimensional systems.

More specifically, in Section 2, we suppose f is a vector field on $\mathbb{R}^2$ which corresponds to a bifurcation point of degree one; that is, the vector field possesses either a saddle-node or a degenerate focus or a degenerate periodic orbit or a homoclinic orbit. The first problem that is considered is the characterization of the flow for a perturbation of this vector field when the perturbation is allowed to be periodic in the independent variable. By using the classical theory of integral manifolds appropriately, we are able to give a complete solution for the saddle-node or degenerate focus. The theory also applies to the degenerate focus even when the perturbations are almost periodic. For the cases corresponding to a degenerate periodic orbit or a homoclinic orbit, only partial results are obtained.

In Section 3, we consider the behavior of the solutions for an autonomous equation in dimension > 2 near an equilibrium point when the linear variational equation has more than two eigenvalues on the imaginary axis. More specifically, we consider an equation in $\mathbb{R}^3$ with two purely imaginary and one zero characteristic root. Using the theory of normal forms and polar coordinates, one can consider this as the periodic perturbation of an autonomous equation in $\mathbb{R}^2$. We first give a complete bifurcation diagram for the autonomous equation under generic hypotheses on the quadratic and cubic terms. Each point on a bifurcation curve is a bifurcation point of degree one. The theory of Section 2 is then applied to obtain a complete description of the problem in $\mathbb{R}^3$. This generalizes results of [5],[9].

We give explicit approximations for the bifurcation curve and show the specific structure of the bifurcation near the homoclinic orbit. Near the Hopf bifurcation, our results could also be obtained from known results on behavior of diffeomorphisms (see, for example, [15],[19]). However, our methods based on the more classical theory of integral manifolds permits the consideration of perturbations which are even almost periodic.

In Section 3, we also discuss an equation in $\mathbb{R}^4$ with two pairs of purely imaginary roots. Using the theory of normal forms and polar coordinates, the problem is reduced to the discussion of an autonomous equation in $\mathbb{R}^2$ perturbed by higher order terms involving two angle variables. This corresponds to a perturbation of the equation in $\mathbb{R}^2$ into $\mathbb{R}^4$. The autonomous equation in $\mathbb{R}^2$ involves no quadratic terms. Under some generic hypotheses on the terms up through fifth degree, we give a bifurcation diagram for the autonomous problem.which has the property that each bifurcation is of degree 1.

In particular, we prove the conjecture in [9] that there is a first integral of the approximate system at the critical values of the parameters where a Hopf bifurcation is conceivable. This has also been recently obtained independently in [6]. From the consideration of higher order terms, we resolve the nature of the bifurcation at this point. Again, using the analysis of Section 2, we mention partial results for the bifurcations in $\mathbb{R}^4$.

1. Summary of known results

If Ω is an open set in $\mathbb{R}^2$, $\Gamma = \partial\Omega$, and $\bar{\Omega} = \Omega \cup \Gamma$, let $\mathscr{X}_2^r = \mathscr{X}_2^r(\bar{\Omega})$ be the set of C^r-vector fields from $\bar{\Omega}$ to $\mathbb{R}^2$ which are transverse to Γ. Two vector fields f,g are equivalent $f \sim g$, if there is a homemorphism on $\bar{\Omega}$ which maps orbits of f onto orbits of g preserving the sense of time. An $f \in \mathscr{X}_2^r$ is structurally stable if there is a neighborhood U of f such that $g \sim f$ if $g \in U$. An $f \in \mathscr{X}_2^r$ is a bifurcation point of degree one if f is not structurally stable and there is a neighborhood U of f such that $g \in U$ implies g is either structurally stable or $g \sim f$.

Theorem 1.1 (see [2],[17]).

An $f \in \mathscr{X}_2^r$ is structurally stable if and only if every equilibrium point and every periodic orbit is hyperbolic and there are no connections between saddle points.

The set of bifurcation points of degree one also is completely classified. The basic result is the following (see [1],[20],[21]).

Theorem 1.2. *A vector field* $f \in \mathfrak{X}_2^r$, $r \geq 3$, *is a bifurcation point of degree* 1 *if and only if there is a neighborhood* W *of* f *and a submanifold* Γ *of codimension one in* W *such that* $W \setminus \Gamma = U_1 \cup U_2$ *where each* $g \in U_j$ *is structurally stable but* $g \not\sim h$ *if* $g \in U_1$, $h \in U_2$. *For* $g \in \Gamma$, *only one of the following situations prevails*:

(i) $g \in \Gamma$ *has an elementary saddle-node at* x_0, *there are no equilibrium points of* g *near* x_0 *if* $g \in U_1$ *and a saddle and node near* x_0 *if* $g \in U_2$.

(ii) $g \in \Gamma$ *has an elementary focus at* x_0, *there is no periodic orbit of* g *near* x_0 *if* $g \in U_1$ *and a periodic orbit near* x_0 *if* $g \in U_2$ - *the generic Hopf bifurcation*.

(iii) $g \in \Gamma$ *has a periodic orbit* γ *which is stable from one side, unstable from the other*, $g \in U_1$ *has no periodic orbit near* γ *and* $g \in U_2$ *has two hyperbolic periodic orbits near* γ.

(iv) $\sigma_0 = \operatorname{tr} \partial f(0)/\partial x \neq 0$, $g \in \Gamma$ *has a homoclinic orbit containing a saddle point* x_0, $g \in U_1$ *has a saddle near* x_0 *and no periodic orbit near* γ, $g \in U_2$ *has a saddle point and a unique hyperbolic periodic orbit near* γ *which coalesce as* $g \to \Gamma$.

Each of the cases (i)-(iv) are shown in Figure 1.

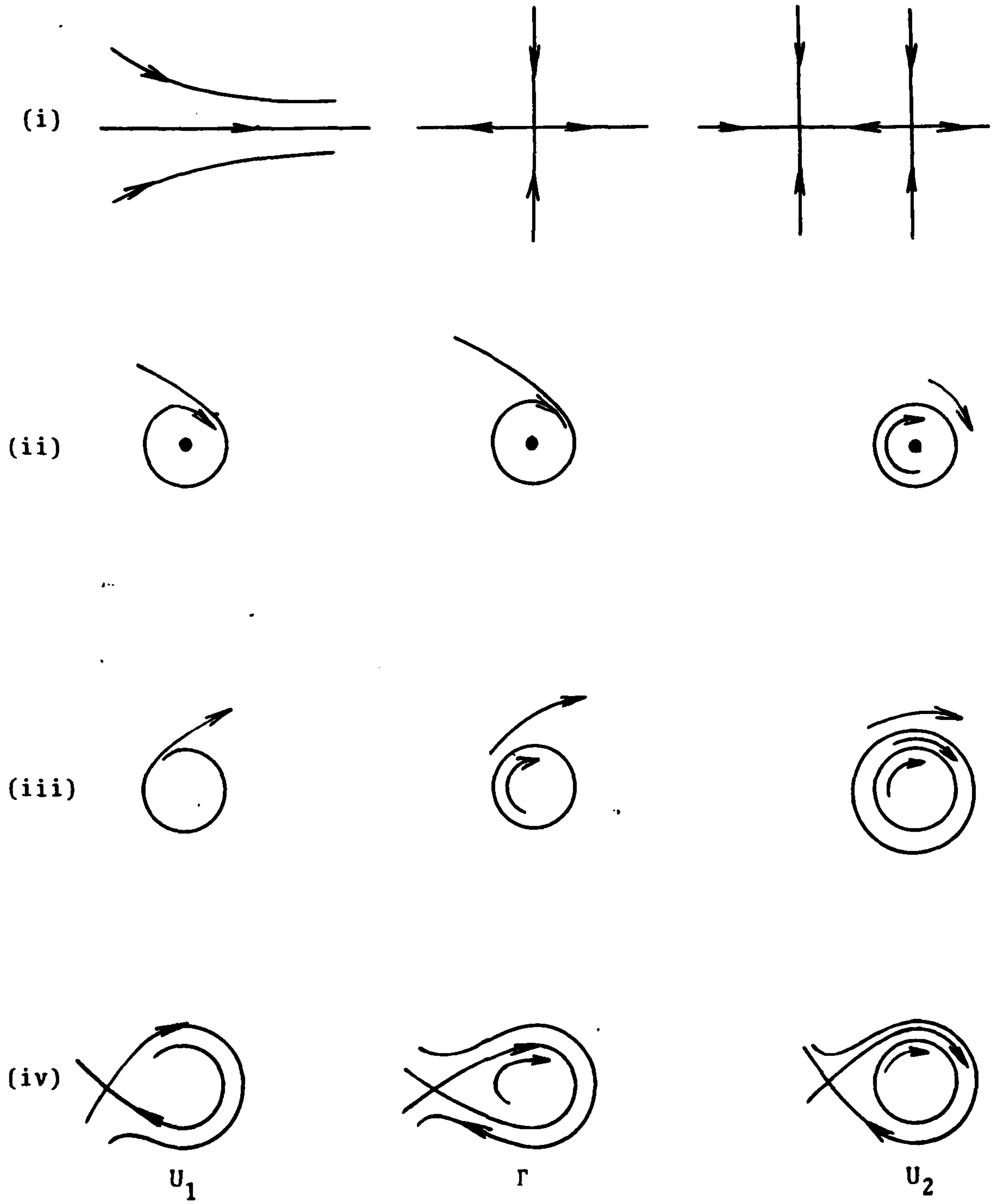

Figure 1.

2. Periodic perturbations.

For vector fields of dimension > 2, the literature on structural stability and bifurcation is extensive, but the results are less complete. We illustrate some of the difficulties by considering perturbation of a planar autonomous system by periodic terms.

Suppose λ,μ are real parameters and consider the system

$$(2.1)_{\lambda,\mu} \qquad \dot{x} = f(x,\lambda) + g(t,x,\mu)$$

where $g(t,x,\mu) = g(t+1,x,\mu), g(t,x,0) = 0$.

We assume that $f(\cdot,0)$ is a bifurcation point of degree one. More precisely, suppose $U = U_1 \cup \Gamma \cup U_2$ as in Theorem 1.2, $f(\cdot,0) \in \Gamma$, $f(\cdot,\lambda) \in U_1$ for $\lambda < 0$, $f(\cdot,\lambda) \in U_2$ for $\lambda > 0$ and the curve $\{f(\cdot,\lambda), \lambda \in \mathbb{R}\}$ is transverse to Γ at $\lambda = 0$.

The vector field $f(\cdot,\lambda)$ is structurally stable for $\lambda \neq 0$ and the manner in which the change is made from one type of structurally stable system to another at $\lambda = 0$ is one of the situations shown in Fig. 1 and described in Theorem 1.2.

By an application of the Implicit Function Theorem, for any $\lambda_0 \in \mathbb{R}$, there is an $\eta(\lambda_0) > 0$, $\nu(\lambda_0) > 0$ such that each hyperbolic equilibrium point of the autonomous equation $(2.1)_{\lambda_0,0}$ is perturbed to a 1-periodic solution of $(2.1)_{\lambda,\mu}$ for $|\lambda-\lambda_0| < \eta(\mu_0)$, $|\mu| < \nu(\lambda_0)$. The classical theory of integral manifolds (see, for example, [7]) implies any hyperbolic periodic orbit is perturbed to a smooth hyperbolic invariant manifold in $\mathbb{R} \times \mathbb{R}^2$ which is like a cylinder whose cross section is one periodic for $|\lambda-\lambda_0| < \eta(\lambda_0)$, $|\mu| < \nu(\lambda_0)$. Identifying the cross section at $t = 0$ with the one at $t = 1$, we obtain a torus. Thus, if $\lambda_0 \neq 0$; that is $f(\cdot,\lambda_0)$ is structurally stable, then the flow for the perturbed equation is understood. Of course, the size of the perturbation, $|\lambda-\lambda_0| < \eta(\lambda_0)$, $|\mu| < \nu(\lambda_0)$, for which this is true depends upon λ_0. The difficulty arises, therefore, in a neighborhood of $\lambda_0 = 0$. Let us discuss this situation in more detail.

Suppose case (i) occurs for $\lambda = 0$, $\mu = 0$. Let p be the saddle node in Fig. 1 and let v be a unit vector in the null space of $\partial f(p,0)/\partial x$. Using the classical method of Liapunov-Schmidt, one obtains a bifurcation function $G(a,\lambda,\mu)$ for $|a|,|\lambda|,|\mu|$ sufficiently small so that $(2.1)_{\lambda,\mu}$ has a 1-periodic solution $x(t)$ in a sufficiently small neighborhood of p if and only if there is an (a,λ,μ) such that $G(a,\lambda,\mu) = 0$ and the constant a is such that $\int_0^1 x(t)dt = p + av$. Furthermore, $G(a,0,0) = \alpha a^2 + O(|a|^2)$ as $|a| \to 0$, where α is a nonzero constant. Using this fact, it is not too difficult to show that a neighborhood U of $(\lambda,\mu) = (0,0)$ can be decomposed as $U = U_1 \cup \Gamma \cup U_2$ where Γ is a smooth curve containing $(0,0)$ such that the period one map of the Eq. $(2.1)_{\lambda,\mu}$ in a neighborhood of p is the same as the one in Fig. 1.1. This gives a complete solution of the problem in case (i).

Suppose $f(\cdot,0)$ satisfies (ii) of Theorem 1.2; that is, the autonomous equation $(\mu=0)$ has a generic Hopf bifurcation. By a transformation of variables, we may assume Eq. $(2.1)_{\lambda,\mu}$ has the form

$$(2.2)_{\lambda,\mu} \qquad \dot{x} = A(\lambda)x + F(x,\lambda) + g(t,x,\mu)$$

with g satisfying the same conditions as before, $F(0,\lambda) = 0$, $\partial F(0,\lambda)/\partial x = 0$ and

$$(2.3) \qquad A(\lambda) = \begin{bmatrix} \alpha_\lambda & \omega_\lambda \\ -\omega_\lambda & \alpha_\lambda \end{bmatrix}, \quad \alpha_0 = 0,\ \omega_0 > 0,$$

and $d\alpha_\lambda/d\lambda|_{\lambda=0} = \alpha_0' \neq 0$.

We make the hypothesis that

$$(2.4) \qquad e^{i\omega_0} \neq 1,$$

that is, the equation $\dot{x} = A(0)x$ has no 1-periodic solutions except $x = 0$.

Hypothesis (2.4) implies there is a neighborhood U of $x = 0$ and a neithborhood V of $(\lambda,\mu) = (0,0)$ such that Eq. $(2.2)_{\lambda,\mu}$ has a unique 1-periodic solution $\phi(\lambda,\mu)$ in U for (λ,μ) in V, $\phi(\lambda,0) = 0$ for all λ. If $x(t) \longmapsto \phi(\lambda,\mu)(t) + x(t)$ then the new equation has the form

$$(2.5)_{\lambda,\mu} \qquad \dot{x} = A(\lambda)x + G(t,x,\lambda,\mu)$$

where $G(t,x,\lambda,\mu)$ is 1-periodic in t,

$$G(t,x,\lambda,\mu) = F(\phi(\lambda,\mu)(t)+x,\lambda) + g(t,\phi(\lambda,\mu)(t)+x,\mu) - F(\phi(\lambda,\mu)(t),\lambda) - g(t,\phi(\lambda,\mu)(t),\mu). \tag{2.6}$$

Thus, $G(t,0,\lambda,\mu) = 0$ for all $t \in \mathbb{R}$, $(\lambda,\mu) \in V$, $G(t,x,\lambda,0) = F(x,\lambda)$.

One now can introduce polar coordinates $x \mapsto (\rho,\theta)$, $x_1 = \rho \cos \theta$, $x_2 = -\rho \sin \theta$, to obtain the equations

$$(2.7)_{\lambda,\mu} \qquad \begin{aligned} \dot{\theta} &= \omega_\lambda + \Theta(t,\theta,\rho,\lambda,\mu) \\ \dot{\rho} &= \alpha_\lambda \rho + R(t,\theta,\rho,\lambda,\mu). \end{aligned}$$

Eq. $(2.7)_{\lambda,\mu}$ must be analyzed for (ρ,λ,μ) in a neighborhood of zero. To use the theory of normal forms or averaging effectively, we make the further nonresonance hypothesis

$$(2.8) \qquad m\omega_0 + n/2\pi \neq 0, \qquad m,n \text{ integers}, \ |m| + |n| \leq 4.$$

Hypothesis (2.8) implies there is a change of variables

$$\theta \mapsto \theta + u(t,\theta,\rho,\lambda,\mu), \quad \rho \mapsto \rho + v(t,\theta,\rho,\lambda,\mu)$$

with u,v periodic in t and θ respectively of period 1 and

$2\pi/\omega_0$, u being $O(|\rho|)$, v being $o(|\rho|)$, for $\lambda,\mu = 0$, such that Eq. $(2.7)_{\lambda,\mu}$ becomes

$$(2.9)_{\lambda,\mu} \quad \begin{aligned} \dot{\theta} &= \omega_\lambda + \beta(\lambda,\mu) + \tilde{\Theta}(t,\theta,\rho,\lambda,\mu) \\ \dot{\rho} &= (\alpha_\lambda+\gamma(\lambda,\mu))\rho + d(\lambda,\mu)\rho^3 + \tilde{R}(t,\theta,\rho,\lambda,\mu) \end{aligned}$$

where $\tilde{\Theta}$ vanishes for $\rho = 0$, $\tilde{R} = o(|\rho|^3)$ as $\rho \to 0$. Furthermore, the properties of G in (2.6) and the fact that $f(\cdot,0)$ corresponds to a generic Hopf bifurcation implies

$$(2.10) \qquad \beta(\lambda,0) = 0,\ \gamma(\lambda,0) = 0,\ d(0,0) \neq 0,\ \alpha_0' d(0,0) < 0,$$

the latter inequality being the case if the picture is the one shown in Fig. 1 for case (ii).

Let C be the curve defined in a neighborhood of $(\lambda,\mu) = 0$ by

$$(2.11) \qquad C = \{(\lambda,\mu) : \alpha_\lambda + \gamma(\lambda,\mu) = 0\}.$$

This curve is well defined since $\alpha_0' \neq 0$ and $\gamma(\lambda,0) = 0$. Introduce new small parameters (ε,δ), $\varepsilon = \alpha_\lambda + \gamma(\lambda,\mu)$, $\delta = \mu$ so that C corresponds to $\varepsilon = 0$.

To be specific in what follows, suppose that $d(0,0) < 0$. In this case, for $(\lambda,\mu) \in C$, that is, $\varepsilon = 0$, the integral manifold $\rho = 0$ of $(2.9)_{\lambda,\mu}$ is uniformly asymptotically stable for δ sufficiently small. For $\varepsilon < 0$, the same is true and one can actually find an $\varepsilon_0 > 0$ and a δ_0 such that this is true for $-\varepsilon_0 \leq \varepsilon \leq 0$, $|\delta| \leq \delta_0$. For each $\varepsilon > 0$, the theory of integral manifolds shows there is an invariant torus which is uniformly asymptotically stable for $|\delta| \leq \delta_0(\varepsilon)$, and attracts everything in a neighborhood of $\rho = 0$ except the line $\rho = 0$. This fact, together with the stability at $\varepsilon = 0$ implies we can find positive ε_0, δ_0 such that all of the above assertions are valid for $|\varepsilon| < \varepsilon_0$, $|\delta| < \delta_0$. Details of the proof are not given here. We note that by introducing new variables $\bar{\rho} = \delta\rho$ and $\bar{\varepsilon} = \delta^2\varepsilon$, we may apply the integral manifold theorem in [7] directly to obtain the desired tori. We may also prove the uniqueness of such tori by using the uniqueness method in [4]. These results are summarized in the following theorem.

Theorem 2.1. Suppose $f(\cdot,0)$ satisfies (ii) of Theorem 1.2 and the equilibrium point of $f(\cdot,\lambda)$ is stable for $\lambda < 0$ and conditions (2.4) and (2.8) are satisfied. Then there is a neighborhood W of $x = 0$, a neighborhood V of $(\lambda,\mu) = 0$ and a smooth curve $C \subset V$ containing $(0,0)$ such that $V = V_1 \cup C \cup V_2$ with V_1, V_2 open connected sets such that the following conclusions hold for all solutions $x(t)$ of $(2.1)_{\lambda,\mu}$ with initial values in W:

(i) For $(\lambda,\mu) \in V_1$, there is a 1-periodic solution $\phi(\lambda,\mu)$ of $(2.1)_{\lambda,\mu}$ with both characteristic multiples less than one and $x(t) - \phi(\lambda,\mu)(t) \to 0$ as $t \to \infty$.

(ii) <u>For</u> $(\lambda,\mu) \in C$, <u>there is a</u> 1-<u>periodic solution</u> $\phi(\lambda,\mu)$ <u>such that both characteristic multipliers are one and the same conclusion as in</u> (i) <u>holds</u>.

(iii) <u>For</u> $(\lambda,\mu) \in V_2$, <u>there is a</u> 1-<u>periodic solution</u> $\phi(\lambda,\mu)$ <u>of</u> $(2.1)_{\lambda,\mu}$ <u>with both characteristic multipliers greater than one and an invariant cylinder in</u> $\mathbb{R}^3$ <u>with</u> 1-<u>periodic cross section which is uniformly asymptotically stable and</u> $x(t)$, $x(0) \neq \phi(\lambda,\mu)(0)$, <u>approaches this cylinder as</u> $t \to \infty$.

If the equilibrium point of $f(\cdot,0)$ is unstable and $f(\cdot,0)$ satisfies (ii) of Theorem 1.2, then a similar result is obtained with obvious changes in the stability statements.

<u>Remark.</u> The above theorem is also true for quasi-periodic g provided the appropriate conditions on the frequencies in terms of roots of unity are assumed. Similar remarks apply when θ is a vector and all functions are periodic in each component of θ.

Now, suppose that $f(\cdot,0)$ satisfies (iii) of Theorem 1.2. This case is much more complicated than the previous one because there is no way to obtain any solution whose qualitative properties are known for all λ,μ (in the previous case, we used strongly the fact that a 1-periodic solution was known to exist for all λ,μ). On the other hand, some results can be obtained, but they are not as complete as in case (ii).

For $\lambda = \mu = 0$, Eq. $(2.1)_{0,0}$ has a unique periodic orbit γ_0 which is asymptotically stable from one side and unstable from the other. Suppose $\gamma_0 = \{x_0(\theta), \theta \in \mathbb{R}\}$ where $x_0(t)$ is a periodic solution of $(2.1)_{0,0}$ of least period $2\pi/\omega_0$, $\omega_0 > 0$. If ρ represents distance from γ_0 along a normal, then one can introduce a local coordinate system around γ_0 which takes $x \mapsto (\theta,\rho)$ to obtain the equations

$$(2.2)_{\lambda,\mu} \qquad \begin{aligned} \dot{\theta} &= 1 + \Theta(t,\theta,\rho,\lambda,\mu) \\ \dot{\rho} &= R(t,\theta,\rho,\lambda,\mu) \end{aligned}$$

where Θ,R vanish for $(\rho,\lambda,\mu) = 0,0,0)$, are 1-periodic in t and $2\pi/\omega_0$ periodic in θ.

Let us suppose that we can average to obtain an equivalent set of equations

$$(2.13)_{\lambda,\mu} \qquad \begin{aligned} \dot{\theta} &= 1 + a\lambda + b\mu + \Theta(t,\theta,\rho,\lambda,\mu) \\ \dot{\rho} &= c\lambda + d\mu + e\rho + f\rho^2 + R(t,\theta,\rho,\lambda,\mu) \end{aligned}$$

where a,b,c,d,e,f are constants, Θ,R for $\rho = 0$ are $O((|\lambda|+|\mu|)^2)$ as $\lambda,\mu \to 0$ and R for $\lambda = \mu = 0$ is $O(\rho^3)$. The hypothesis that $f(\cdot,0)$ satisfies (iii) of Theorem 1.2 implies $cf < 0$, $e = 0$. Since $c \neq 0$, let us introduce new parameters (δ,ε) as $\delta = \mu$, $\varepsilon = c\lambda + d\mu$ to obtain the equations

$$(2.14)_{\varepsilon,\delta} \qquad \begin{aligned} \dot{\theta} &= 1 + a'\delta + b'\varepsilon + \Theta'(t,\theta,\rho,\varepsilon,\delta) \\ \dot{\rho} &= \varepsilon + f\rho^2 + R'(t,\theta,\rho,\varepsilon,\delta) \end{aligned}$$

with Θ', R' having the same type of order relations as before.

For any ε small enough such that $\varepsilon f > 0$, it is shown there is a $\delta_0(\varepsilon) > 0$ such that no solution of $(2.14)_{\varepsilon,\delta}$ remains in a neighborhood of $\rho = 0$ for all t if $0 \le \delta < \delta_0(\varepsilon)$. This is the analogue of $\lambda < 0$ and the picture in Fig. 1 for the period one map. If $\varepsilon f < 0$, then the theory of integral manifolds shows there is a $\delta_0(\varepsilon) > 0$ such that there are two invariant cylinders $T^1_{\varepsilon,\delta}$, $T^2_{\varepsilon,\delta}$ of $(2.14)_{\varepsilon,\delta}$, $0 \le \delta < \delta_0(\varepsilon)$, with periodic cross section; that is, two invariant tori $T^1_{\varepsilon,\delta}$, $T^2_{\varepsilon,\delta}$. One of these tori is completely unstable and the other is uniformly asymptotically stable. Furthermore, any solution of $(2.14)_{\varepsilon,\delta}$ which remains in a sufficiently small neighborhood of $\rho = 0$ for $t \ge 0$ $(t \le 0)$ must approach either $T^1_{\varepsilon,\delta}$ or $T^2_{\varepsilon,\delta}$ as $t \to \infty (t \to -\infty)$. Thus, we have complete knowledge of the solutions in a subset of a neighborhood of $(\varepsilon,\delta) = (0,0)$. The subset is depicted in Fig. 2.

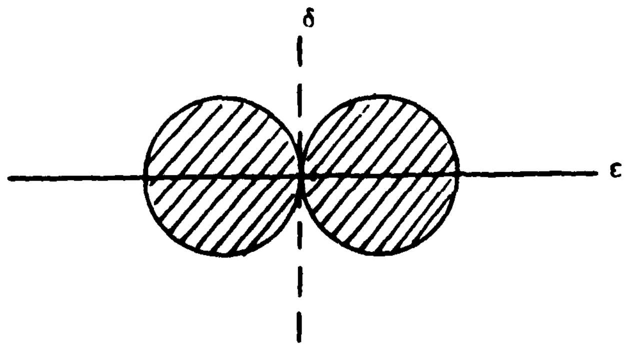

Figure 2.

As $\varepsilon \to 0$, the number $\delta_0(\varepsilon)$ may also approach zero. The reason for this fact is that the invariant tori $T^1_{\varepsilon,\delta}$, $T^2_{\varepsilon,\delta}$ for a fixed ε may fail to exist as smooth invariant sets for some small δ. This is a general fact in differential equations. Given a smooth hyperbolic invariant torus for a differential equation, the existence of a smooth invariant torus for a small perturbation of this equation generally requires that the strength of the hyperbolicity is greater than the rate of attraction of orbits on the torus. If this is not the case, the invariant sets develop cusps at rational rotation numbers. For the sets $T^1_{\varepsilon,\delta}$, $T^2_{\varepsilon,\delta}$ above, the strength of the hyperbolicity approaches zero as $\varepsilon \to 0$ for any fixed δ. Thus, it is conceivable that the sets touch for some ε without coalescing in a uniform manner. This is why we cannot complete the picture in Fig. 2. The behavior in the unshaded region will be extremely complicated and no one has made any progress toward an explanation. The classical theory of integral manifolds gives no information.

Suppose now that $f(\cdot,0)$ satisfies condition (iv) of Theorem 1.2. This situation has been discussed much more in the literature because so many new phenomena occur. If p is the saddle point of the homoclinic orbit γ of $(2.1)_{0,0}$, then the Implicit Function Theorem gives a 1-periodic solution $\phi(\lambda,\mu)$ of $(2.1)_{\lambda,\mu}$ for λ,μ small, $\phi(0,0) = p$, and $P_{\lambda,\mu} \overset{\text{def}}{=} \phi(\lambda,\mu)(0)$ will be a saddle point for the period one map $\pi = \pi_{\lambda,\mu}$. Let $W^s_{\lambda,\mu}$, $W^u_{\lambda,\mu}$ be the stable and unstable manifolds of $P_{\lambda,\mu}$ as a fixed point of π.

A point q is homoclinic to $P_{\lambda,\mu}$ if $q \in W^s_{\lambda,\mu} \cap W^u_{\lambda,\mu}$. It is transverse homoclinic to $P_{\lambda,\mu}$ if it is homoclinic to $P_{\lambda,\mu}$ and, in addition, $W^s_{\lambda,\mu}$, $W^u_{\lambda,\mu}$ are transversal at q. For $\lambda = \mu = 0$, any point $q \in \gamma$ is homoclinic to $p_0 = x_0$, but it is not transverse homoclinic. Under a small perturbation in an appropriate direction, one would expect that $W^s_{\lambda,\mu}$ and $W^u_{\lambda,\mu}$ intersect transversally at some point q close to γ. The behavior of the solutions of $(2.1)_{\lambda,\mu}$ is then very complicated near γ. In fact, one can show there are infinitely many distinct periodic solutions. Also, there is an invariant set of π^k for some integer k for which the flow is equivalent to the shift automorphism on two symbols (see [14]).

It has also been shown that, in a neighborhood of the bifurcation point, there are generically in g infinitely many hyperbolic sources (or sinks) (see [16]). This suggests that the bifurcation to transverse homoclinic points might be related to some other types of successive bifurcations of periodic orbits of longer and longer period. The existence of these successive bifurcations have not been studied when $f(\cdot,0)$ satisfies (iv) in Theorem 1.2.

For the case where $f(\cdot,0)$ is Hamiltonian, successive bifurcations appear to be the general rule as indicated by the results in [3]. Let us summarize these results for a special case. Consider the equation

$$\dot{x} = y,$$

$$(2.15)_{\lambda,\mu} \qquad \dot{y} = x - x^2 - \lambda y + \mu f(t)$$

where λ,μ are small parameters, f is continuous and $f(t+1) = f(t)$. Eq. $(2.15)_{0,0}$ has a homoclinic orbit γ, but it does not satisfy (iv) of Theorem 1.2 since the divergence of the vector field at the saddle point $(0,0)$ is zero. Let $\gamma = \{(0,0)\} \cup \{(p(t),\dot{p}(t)), -\infty < t < \infty\}$ where $(p(t),\dot{p}(t))$ is a solution of $(2.15)_{0,0}$ which approaches zero as $t \to \pm\infty$. We may normalize p so that $\dot{p}(0) = 0$.

Let

$$(2.16) \qquad h(\phi) = \int_{-\infty}^{\infty} \dot{p}(t)f(t-\phi)dt \Big/ \int_{-\infty}^{\infty} \dot{p}^2(t)dt$$

and suppose that every extreme point of h is a strict local maximimum or local minimum. Let $\phi_M,\phi_m \in [0,1)$ correspond respectively to the absolute maximum and minimum of h. We now state the following result from [3]. Parts (i), (ii) were also essentially given in [13].

<u>Theorem 2.2.</u> <u>Under the above hypotheses, there is a neighborhood</u> U <u>of</u> γ <u>in</u> $\mathbb{R}^2$, <u>a neighborhood</u> V <u>of</u> $(\lambda,\mu) = (0,0)$ <u>in</u> $\mathbb{R}^2$, <u>an integer</u> $k_0 > 0$ <u>and curves</u> C_M^k, C_m^k <u>in</u> V, $k = k_0,k_0+1,\ldots,\infty$ <u>(including</u> ∞) <u>which have parametric representations with twice</u>

continuously differentiable functions, each C_M^k, C_m^k contains $(0,0)$, $C_M^k \to C_M^\infty$, $C_m^k \to C_m^\infty$ as $k \to \infty$ uniformly, $C_M^\infty[C_m^\infty]$ is tangent to the curve defined by $\lambda = h(\phi_M)\mu$ $[\lambda = h(\phi_m)\mu]$ at $(\lambda,\mu) = (0,0)$ and for each integer $k \in [k_0,\infty]$, the set $V \setminus (C_M^k \cup C_m^k) = S_1^k \cup S_2^k$, S_1^k, S_2^k open, disjoint, and the following conclusions hold for Eq. $(2.15)_{\lambda,\mu}$

(i) $\lambda \in S_1^\infty$ implies no homoclinic point exists in U

(ii) $\lambda \in S_2^\infty$ implies a transverse homoclinic point exists in S_2^∞

(iii) If $k < \infty$, $\lambda \in S_1^k$ implies no subharmonic solution in U

(iv) If $k < \infty$, $\lambda \in S_2^k$ implies there are at least $2k$ hyperbolic subharmonics in U.

(v) If $\lambda \in \bigcap_{k \geq k_0} S_2^k$, then there are infinitely many hyperbolic nodes or foci in U corresponding to fixed points of iterates of the period one map.

The above theorem shows, in particular, the following. If (λ,μ) is a point of bifurcation to transverse homoclinic orbits; that is, $(\lambda,\mu) \in C_M^\infty$ or C_m^∞, and W is any neighborhood of this point, then there are infinitely many subharmonic bifurcations in W.

Let us outline the proof. In $(2.15)_{\lambda,\mu}$, if $x(\tau) = p(\tau+\phi) + z(\tau+\phi)$, $\tau + \phi = t$, then

$$\ddot{z} + a(t)z = F(t,z,\dot{z},\lambda,\mu,\phi) \tag{2.17}$$

where $a(t) = -1 + 2p(t)$, the derivative of $-x + x^2$ evaluated at $p(t)$. The function F is given by

$$F = -\lambda\dot{z} - \lambda\dot{p} + \mu f(t-\phi) - z^2.$$

The first step of the proof is to obtain conditions for the existence of a homoclinic orbit. This is equivalent to finding a solution z of (2.17) which approaches zero as $t \to \pm\infty$. We, therefore, need an analogue of the Fredholm alternative for bounded solutions on $\mathbb{R}$ of the nonhomogeneous equation

$$\ddot{z} + a(t)z = F(t)$$

where $F(t)$ is bounded and continuous on $\mathbb{R}$. A careful analysis of the variation of constants formula shows that such a solution exists if and only if

$$\int_{-\infty}^{\infty} \dot{p}(t)F(t)dt = 0,$$

If this condition is satisfied, then the solution is unique.

After having this Fredholm alternative, one can adapt the reduction principle of Liapunov-Schmidt or the method of alternative problems to this situation to obtain a bifurcation function $G(\phi,\lambda,\mu)$ defined for $\phi \in \mathbb{R}$, (λ,μ) in a neighborhood of $(0,0)$, which has the property that there is a homoclinic orbit in a small neighborhood of γ, if and only if $G(\phi,\lambda,\mu) = 0$. Furthermore, it is transverse homoclinic if and only if $\partial G(\phi,\lambda,\mu)/\partial\phi \neq 0$. To solve this equation, one observes that

$$G(\phi,\lambda,\mu) = -\lambda + \mu h(\phi) + O((|\lambda|+|\mu|)^2)$$

as $\lambda,\mu \to 0$ where $h(\phi)$ is defined in (2.16). The proof of parts (i) and (ii) in the theorem are now supplied without much difficulty.

To prove the other parts, one uses the results in [8] for obtaining the curves of bifurcation to subharmonics of order k coming from the orbit of period k of $(2.15)_{0,0}$. One then shows by a nontrivial argument that the conditions on $h(\alpha)$ imply the ones in [8] uniformly in k. This will complete the proof (see [3] for details).

3. Two purely imaginary and one zero root.

In this section, we consider a differential equation in $\mathbb{R}^3$ for which the linear variational equation near the equilibrium point zero has two purely imaginary eigenvalues and one zero eigenvalue. To simplify the situation, it will be assumed also that a certain type of symmetry prevails. More specifically, consider the equation

$$\begin{aligned}\dot{x} &= A(\lambda)x + f(x,y) \\ \dot{y} &= \beta y + g(x,y)\end{aligned} \tag{3.1}$$

where λ, β are small real parameters, $x \in \mathbb{R}^2$, $y \in \mathbb{R}$, f, g are C^4-functions,

$$A(\lambda) = \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix},$$

(3.2) $$f(x,y) = O(|x|(|x|+|y|))$$

$$g(x,y) = O((|x|+|y|)^2)$$

as $|x|, |y| \to 0$. The hypothesis on f implies that the symmetry condition

(3.3) $$f(0,y) = 0$$

is satisfied.

Since $f(0,y) = 0$, it is legitimate to introduce polar coordinates for $x = (x_1, x_2)$ as $x_1 = r \cos \theta$, $x_2 = -r \sin \theta$. If this is done and t is replaced by θ, one obtains the equations

(3.4) $$\begin{aligned} \dot{\rho} &= \lambda\rho + R(\theta,\rho,y,\lambda,\beta) \\ \dot{y} &= \beta y + Y(\theta,\rho,y,\lambda,\beta) \end{aligned}$$

where $R = O(|\rho|(|\rho|+|y|))$, $Y = O((|\rho|+|y|)^2)$ as $\rho, y \to 0$.

The problem is to determine the behavior of the solutions of (3.4) in a neighborhood of $(\rho,y) = (0,0)$ for (λ,β) in a neighborhood of $(0,0)$. To discuss (3.4), one can use the theory of normal forms (or averaging) to transform (3.4) to a system

$$\begin{aligned} \dot{\rho} &= \rho(\lambda+ay) + O(|\rho|(\rho^2+y^2)) \\ \dot{y} &= \beta y + by^2 + c\rho^2 + O((|\rho|+|y|)^3) \end{aligned} \tag{3.5}$$

as $\rho, y \to 0$. The generic situation is to have a,b,c nonzero constants.

To simplify the computations, and also to consider the most interesting case, we suppose $a = 2$, $b = c = -1$ so that the equations are

$$\begin{aligned} \dot{\rho} &= \rho(\lambda+2y) + O(|\rho|(\rho^2+y^2)) \\ \dot{y} &= \beta y - y^2 - \rho^2 + O((|\rho|+|y|)^3) \end{aligned} \tag{3.6}$$

as $\rho, y \to 0$.

It is convenient to introduce the change of variables $y \mapsto \beta/2 + y$, $\lambda + \beta \mapsto \alpha$, to obtain the more symmetric form

$$\begin{aligned} \dot{\rho} &= \rho(\alpha+2y) + O(|\rho|(\rho^2+(\beta/2+y)^2) \\ \dot{y} &= \frac{\beta^2}{4} - y^2 - \rho^2 + O((|\rho|+\beta/2+y)^3) \end{aligned} \tag{3.7}$$

as $\rho, \beta, y \to 0$.

If we perform the scalings

$$(3.8) \qquad \rho \to \varepsilon\rho, \quad y \to \varepsilon y, \quad \beta \to \varepsilon, \quad \alpha \to \varepsilon\alpha, \quad t \to \varepsilon^{-1}t$$

the new equations become

$$(3.9) \qquad \begin{aligned} \dot{\rho} &= 2\rho y + \alpha\rho + O(|\varepsilon|) \\ \dot{y} &= \frac{1}{4} - y^2 - \rho^2 + O(|\varepsilon|) \end{aligned}$$

as $\varepsilon \to 0$.

Equations (3.9) must be discussed for all $\rho \geq 0$, $y \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and ε in a neighborhood of $\varepsilon = 0$.

For $\alpha = 0$, $\varepsilon = 0$, the function

$$(3.10) \qquad V(\rho,y) = \frac{1}{4}\rho - \rho y^2 - \frac{\rho^3}{3}$$

is a first integral of (3.9). Since the Jacobian of the vector field in (3.9) has trace α, it follows that there can be no periodic orbit of (3.9) unless $\alpha = 0$. This remark makes much of the discussion of (3.9) very simple for $\alpha \neq 0$, $\varepsilon = 0$. In fact, the topological structure of the flow is determined by the qualitative properties of the equilibrium points.

The equilibrium points of (3.9) for $\varepsilon = 0$ are $\rho = 0$, $y = \pm \frac{1}{2}$ for all α and the point $y = -\alpha/2$, $\rho^2 = (1-\alpha^2)/4$ for

$\alpha^2 \leq 1$. For $\alpha^2 \neq 1$, the points $\rho = 0$, $y = \pm 1/2$ are hyperbolic saddles or nodes. In fact, for $\alpha > -1$ $(\alpha < -1)$, the point $(1/2,0)$ is a saddle (stable node). For $\alpha > 1(<1)$, the point $(-1/2,0)$ is a saddle (unstable node). For $\alpha > 0$ (<0), the point $(-\alpha/2,(1-\alpha^2)^{1/2}/2)$ is a stable (unstable) focus. For $\alpha = 1$, the point $(-1/2,0)$ is a saddle-node. For $\alpha = -1$, the point $(1/2,0)$ is a saddle-node. These facts imply that the phase portrait for (3.9) for $\alpha \neq 0$, $\varepsilon = 0$ are the ones shown in Fig. 3. The only bifurcation points are when $\alpha = \pm 1$ and these are of saddle-node type. For $\alpha = 0$, $\varepsilon = 0$, the function V in (3.10) is a first integral and the phase portrait is determined from the level curves of this function shown in Fig. 3.

As remarked in the previous section, for each $\alpha \neq 0$, there is an $\varepsilon_0(\alpha)$ such that the same portraits are valid for the period map (remember the terms in ε are periodic in Θ) of the complete equations (3.9) for $0 \leq \varepsilon \leq \varepsilon_0(\alpha)$. This gives a uniform estimate in ε as long as α remains in a compact set. To obtain a uniform estimate for α large, we return to the original equation (3.7) and introduce the scaling

$$\rho \mapsto \varepsilon\rho, \quad y \mapsto \varepsilon y, \quad \alpha \mapsto \varepsilon, \quad \beta \mapsto \varepsilon\beta, \quad t \mapsto \varepsilon^{-1}t$$

to obtain the equations

$$\dot{\rho} = 2\rho y + \rho + O(|\varepsilon|)$$

$$\dot{y} = \frac{\beta^2}{4} - y^2 - \rho^2 .$$

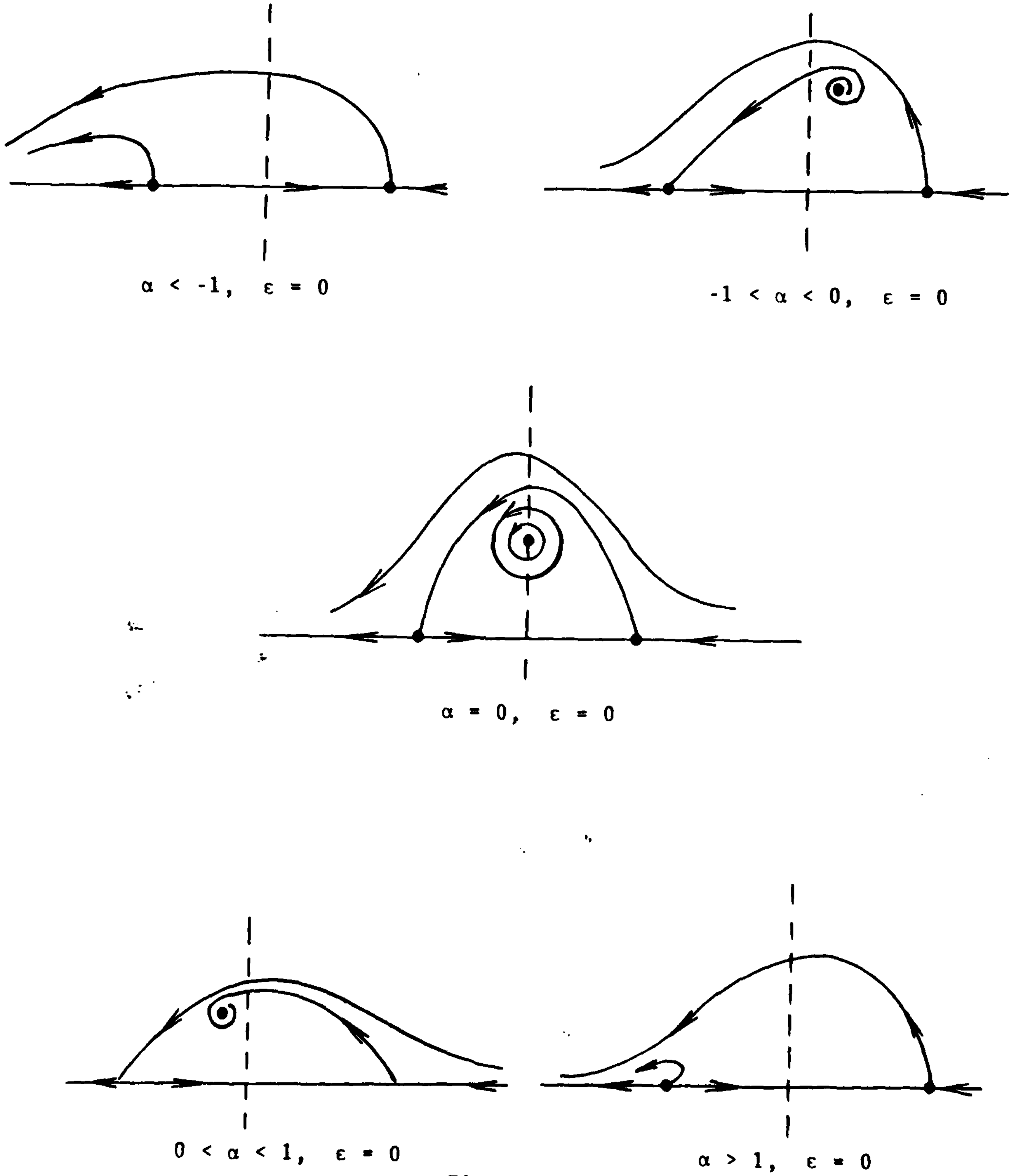
$\alpha < -1, \quad \varepsilon = 0$
$-1 < \alpha < 0, \quad \varepsilon = 0$
$\alpha = 0, \quad \varepsilon = 0$
$0 < \alpha < 1, \quad \varepsilon = 0$
$\alpha > 1, \quad \varepsilon = 0$

Figure 3.

In this equation, we need to discuss only the case where ε is near zero and β varies in a neighborhood of zero because the case for large β is the same as the previous one with the scaling (3.8). For $\varepsilon = 0$, $\beta = 0$, the equations have only one equilibrium point $\rho = 0$, $y = 0$ and this is a saddle-node. The analysis in the previous section applies to this case.

This shows that the only case remaining to discuss in Eq. (3.9) is $\alpha = 0$, $\varepsilon = 0$. The manner in which the phase portrait in Fig. 3 changes as we vary (α,ε) in a neighborhood of $(0,0)$ cannot be determined with specific knowledge of the terms in order ε. Thus suppose system (3.6) is written as

$$\begin{aligned} \dot{\rho} &= \rho(\lambda+2y+d\rho^2+ey^2) + O((|\rho|+|y|)^4) \\ \dot{y} &= \beta y - y^2 - \rho^2 + f\rho^2 y + gy^3 + O((|\rho|+|y|)^4) \end{aligned} \tag{3.11}$$

as $\rho, y \to 0$.

The final result depends on all of the constants d,e,f,g. To simplify the computations, we suppose $d = 1$, $e = f = g = 0$. The general case is treated in the same way although more computations are involved. Thus, our system is

$$\begin{aligned} \dot{\rho} &= \rho(\lambda+2y+\rho^2) + O((|\rho|+|y|)^4) \\ \dot{y} &= \beta y - y^2 - \rho^2 + O((|\rho|+|y|)^4). \end{aligned} \tag{3.12}$$

Making the same transformations as before and using the scaling (3.8), one obtains the Equation (3.9) in the more explicit form

$$\dot{\rho} = 2\rho y + \alpha\rho + \varepsilon\rho^3 + O(|\varepsilon|^2)$$

(3.13)

$$\dot{y} = \frac{1}{4} - y^2 - \rho^2 \qquad + O(|\varepsilon|^2)$$

as $\varepsilon \to 0$.

Our first objective is to analyze Eq. (3.13) when the ε^2 terms are neglected; that is, the equation

$$\dot{\rho} = 2\rho y + \alpha\rho + \varepsilon\rho^3$$

(3.14)

$$\dot{y} = \frac{1}{4} - y^2 - \rho^2 \quad .$$

For $\alpha = \varepsilon = 0$, the phase portrait is given in Fig. 3. Let $(\rho_0(t), y_0(t))$, $\rho_0(t) > 0$, be the solution of (3.14) for $\alpha = \varepsilon = 0$ describing the heteroclinic orbit; that is, $\rho_0(t) \to 0$ as $t \to \pm\infty$, $y_0(t) \to -1/2$ as $t \to \infty$, $y_0(t) \to 1/2$ as $t \to -\infty$. We may assume $y_0(0) = 0$. In a neighborhood of $\alpha = \varepsilon = 0$, there will be a curve with the property that Eq. (3.14) has a heteroclinic orbit joining $(-1/2, 0)$, $(1/2, 0)$ for each (α, ε) belonging to this curve. The determination of this curve follows along the same ideas as in the case (iv) of the previous section as an application of the method of Liapunov-Schmidt. One constructs a scalar function $G(\alpha,\varepsilon)$, $G(0,0) = 0$, such that there is a heteroclinic orbit if and only if $G(\alpha,\varepsilon) = 0$. Furthermore,

$$G(\alpha,\varepsilon) = \alpha\int_{-\infty}^{\infty} \rho_0^2(t)dt + \varepsilon \int_{-\infty}^{\infty} \rho_0^4(t)dt + O((|\alpha|+|\varepsilon|)^2)$$

as $\alpha,\varepsilon \to 0$. The Implicit Function Theorem implies there is a unique solution $\alpha = \alpha^*(\varepsilon)$ of $G(\alpha,\varepsilon) = 0$ for ε sufficiently small, $\alpha^*(0) = 0$ and

$$\delta \overset{\text{def}}{=} \frac{d\alpha^*(0)}{d\varepsilon} = -\frac{\int_{-\infty}^{\infty} \rho_0^4(t)dt}{\int_{-\infty}^{\infty} \rho_0^2(t)dt} = -\frac{1}{2} \quad .$$

The specific value of δ *as* $-\frac{1}{2}$ *is obtained in the following way.* Since

$$\rho_0^2 dy_0 = (\tfrac{1}{4}\rho_0^2 - \rho_0^2 y_0^2 - \rho_0^4)dt$$

$$-\int_{-\infty}^{\infty} \rho_0^4 dt = \delta\int_{-\infty}^{\infty} \rho_0^2 dt$$

$$0 = \rho_0 V(\rho_0, y_0) = \tfrac{1}{4}\rho_0^2 - \rho_0^2 y_0^2 - \tfrac{1}{3}\rho_0^4$$

we have

$$\int_{\frac{1}{2}}^{-\frac{1}{2}} \rho_0^2 dy_0 = (\tfrac{1}{4}+\delta)\int_{-\infty}^{\infty} \rho_0^2 dt - \int_{-\infty}^{\infty} \rho_0^2 y_0^2 dt$$

$$= \frac{2\delta}{3} \int_{-\infty}^{\infty} \rho_0^2 dt$$

$$= \frac{4\delta}{3} \int_0^{\infty} \rho_0^2 dt$$

$$= \frac{2\delta}{3} \int_0^{\rho_0(0)} \frac{\rho_0}{y_0}\, d\rho_0$$

from $d\rho_0 = 2\rho_0 y_0 dt$ and the symmetry in the equation. Since $V(\rho_0(t), y_0(t)) = 0$, we have $\rho_0^2(0) = 3/4$, $\rho_0^2 = 3(1/4 - y_0^2)$, $y_0 = (1/4 - \rho_0^2/3)^{1/2}$. Using these relations we have, for $\theta = \rho_0^2$

$$2\int_0^{\rho_0(0)} \frac{\rho_0}{y_0}\, d\rho_0 = \int_0^{3/4} (\tfrac{1}{4} - \tfrac{\theta}{3})^{-1/2} d\theta = 3$$

$$\int_{-1/2}^{1/2} \rho_0^2 dy_0 = 3\int_{-1/2}^{1/2} (\tfrac{1}{4} - y^2) dy = \tfrac{1}{2} \quad .$$

Thus, $\delta = -1/2$.

The function $\alpha = \alpha^*(\varepsilon)$ satisfying $G(\alpha^*(\varepsilon), \varepsilon) = 0$ for $|\varepsilon| < \varepsilon_0$ has the property that there is a heteroclinic orbit of Eq. (2.14) for $(\alpha, \varepsilon) = (\alpha^*(\varepsilon), \varepsilon)$.

Near the value $\alpha = 0$, $\varepsilon = 0$, there is also a curve where a Hopf bifurcation occurs. For $\alpha = 0$, $\varepsilon = 0$, Eq. (3.14) has an equilibrium point $y = 0$, $\rho = 1/2$. The Implicit Function Theorem implies there is an equilibrium close to this one for

(α,ε) small and it is given approximately by $y_0 = -(\alpha+\varepsilon/4)/2$, $\rho_0 = 1/2 + y_0$. Analyzing the stability properties of this solution, one sees that it has eigenvalues on the imaginary axis along a curve approximately given by $\alpha = -3\varepsilon/4$.

We next analyze the periodic orbits of (3.14) for (α,ε) small. Any such orbit must be close to one of the periodic orbits for $(\alpha,\varepsilon) = (0,0)$. For $(\alpha,\varepsilon) \neq (0,0)$ let $y(t)$, $\rho(t)$ be a periodic solution of (3.14) of period T, normalized so that $y(0) = 0$. Following the procedure in [3], one can determine necessary and sufficient conditions on (α,ε) in order that (3.14) has a periodic orbit of period T near the orbit γ_T defined by $y(t)$, $\rho(t)$. This is an application of the Fredholm alternative and the method of Liapunov-Schmidt to obtain a bifurcation function $G_T(\alpha,\varepsilon)$ with the property that there is such a period orbit if and only if $G_T(\alpha,\varepsilon) = 0$. The function $G_T(\alpha,\varepsilon)$ has the form

$$G_T(\alpha,\varepsilon) = \alpha\int_{-T/2}^{T/2} \rho^2(t)dt + \varepsilon\int_{-T/2}^{T/2} \rho^4(t)dt + O((|\alpha|+|\varepsilon|)^2) \tag{3.15}$$

as $(\alpha,\varepsilon) \to (0,0)$.

The Implicit Function Theorem implies there is a unique solution $\alpha = \alpha^*(\varepsilon,T)$ of (3.15) for ε sufficiently small, $\alpha^*(0,T) = 0$ and

$$\delta_T \overset{\text{def}}{=} \frac{d\alpha^*(0,T)}{d\varepsilon} = -\frac{\int_0^T \rho^4(t)dt}{\int_0^T \rho^2(t)dt} .$$

Using the first integral again, it is not difficult to see that the period T is a strictly decreasing function of $\rho(0)$ approaching 2π as $\rho \to 1/2$. This implies there is a unique periodic orbit of period T on each of the curves defined by $\alpha = \alpha^*(\varepsilon,T)$ above.

Using the differential equation and the first integral V, one can show that

$$\int_{\gamma_T} \rho^2 dy = \frac{2}{3}\rho_T \int_{\rho(0)}^{\rho(T/2)} \frac{r^2}{[a_T T + r^4/3 - r^2/4]^{1/2}} dr$$

where $a_T = V(\rho(0),y(0)) = \rho(0)/4 - \rho^3(0)/12$.

As $T \to \infty$, it is not difficult to see that $a_T T \to 0$, $\delta_T \to \delta$ as $T \to \infty$ where $\delta = -1/2$ is the number computed before representing the slope of the curve defining the homoclinic orbit. As $T \to 2\pi$, $a_T \to 11/96$, $\delta_T \to -3/4$, the slope of the curve defining the Hopf bifurcation. Consequently, there is a unique periodic orbit of (3.14) between these two extreme curves.

We have, therefore, obtained a complete analysis of the behavior of the solutions of (3.14) for all $\alpha \in \mathbb{R}$, ε small.

If we return to the original coordinate system in (3.12), we can state the following result.

Theorem 3.1. There is a neighborhood U of $(\rho, y) = (0,0)$ and a neighborhood V of $(\lambda, \beta) = 0$ such that the neighborhood V is divided into regions as shown in Fig. 4 such that the flow for Eq. (3.14) in each region is the one depicted in Fig. 5. The curves $\Gamma_1, \Gamma_2, \Gamma_3$ are given approximately by

$$\Gamma_1 : \lambda \sim -\beta - \beta^2/2$$

$$\Gamma_2 : \lambda \sim -\beta - 3\beta^2/4$$

$$\Gamma_3 : \lambda \sim -2\beta$$

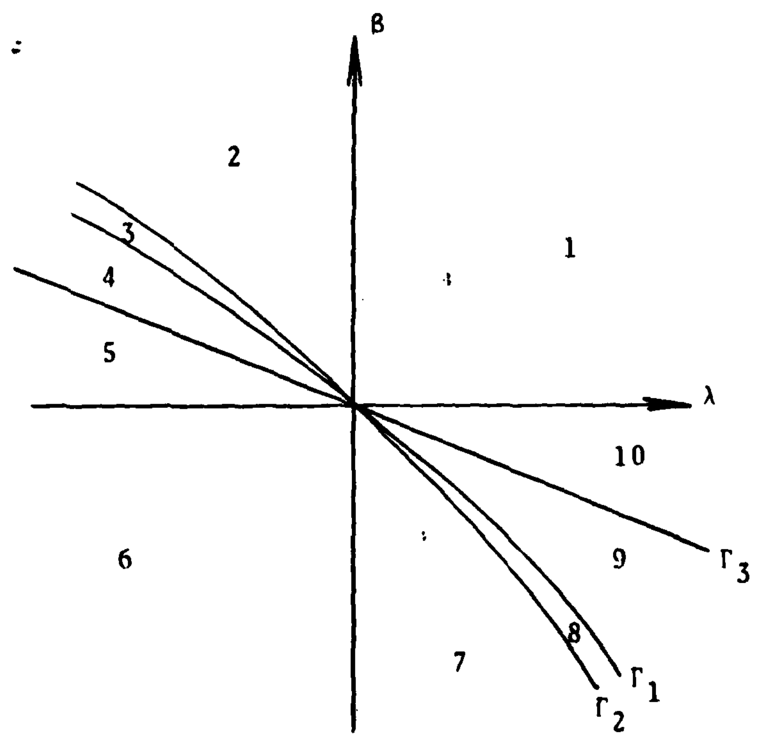

Figure 4.

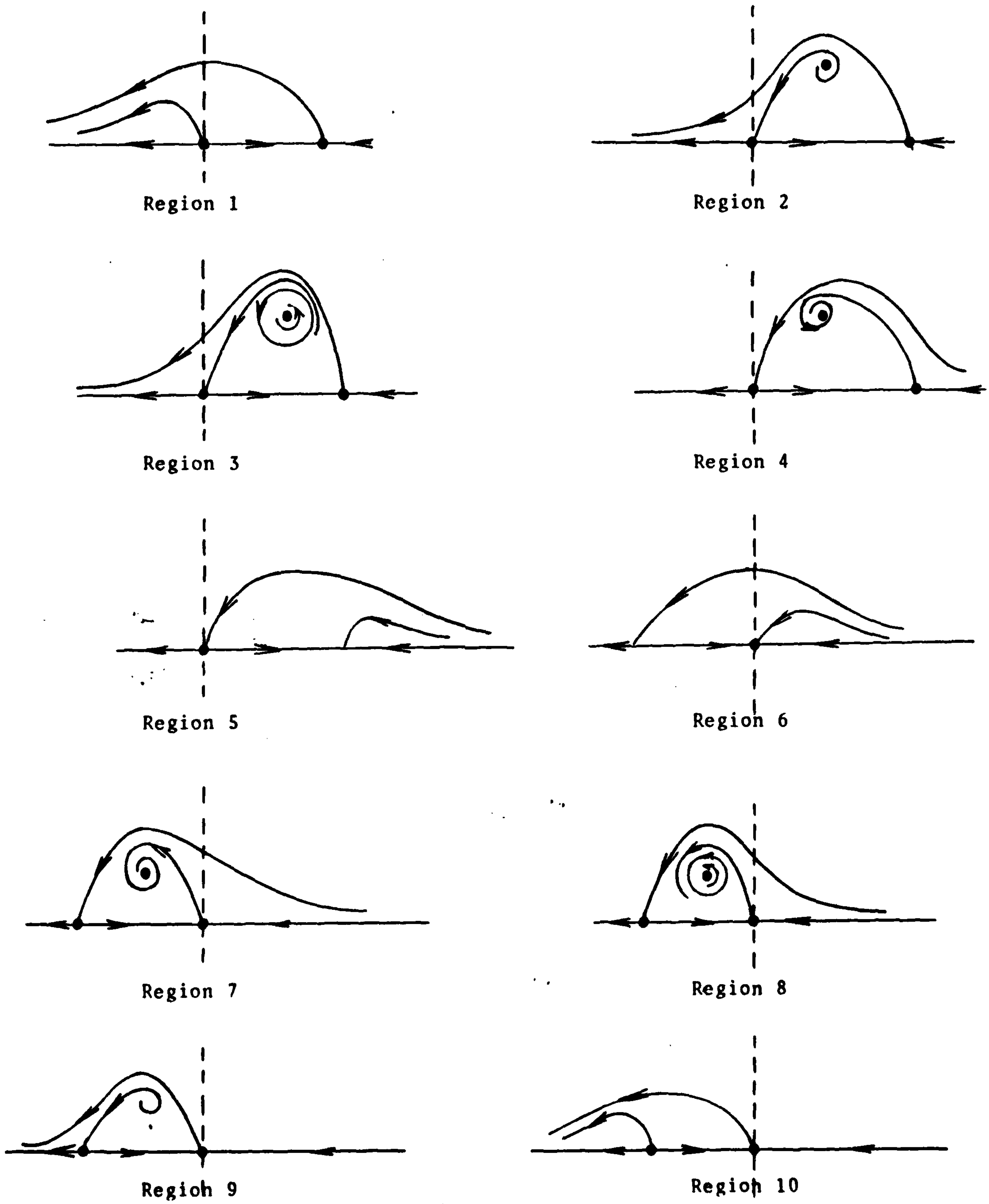
Region 1
Region 2
Region 3
Region 4
Region 5
Region 6
Region 7
Region 8
Region 9
Region 10

Figure 5.

All curves in Fig. 4 correspond to saddle-node bifurcations except Γ_1 and Γ_2. Each point on Γ_2 corresponds to a generic Hopf bifurcation and each point on Γ_1 corresponds to bifurcation through a homoclinic orbit.

We remark that the same conclusions as in Theorem 3.1 will be valid for (3.11) for most values of the modal parameters d,e,f,g. Only the terms in β^2 of the curves Γ_1,Γ_2 will be changed. There will be a submanifold of codimension 2 where the complete description of the flow will require the terms of order higher than in (3.11).

Theorem 3.1 gives a complete description of the flow for the autonomous part of Eq. (3.11) obtained by neglecting terms of order >3. These higher order terms are periodic in the independent variable. Thus, we have a problem similar to the one described in the previous section. The results there and especially Theorem 2.1 gives a description of the flow for the complete equations except in a neighborhood of the curve Γ_1 of bifurcation through a homoclinic orbit. The periodic perturbation near points on this curve can change the structure of the flow in a significant way as we have seen in Section 2. Points on this curve are analyzed in the same manner as in Section 2 in the following way.

Suppose we consider the scaled equations (3.13). Applying the method of Section 2 to obtain the curves in the α,ε plane for which there is a homoclinic orbit, we choose a point (α_0,ε_0).

corresponding to a point on Γ_1, introduce a phase shift ϕ along the homoclinic orbit corresponding to $(\alpha_0, \varepsilon_0)$. If $\alpha = \alpha_0 + \nu$, $\varepsilon = \varepsilon_0 + \mu$, then one can obtain a bifurcation function $\tilde{G}(\phi, \nu, \mu)$ for $\phi \in \mathbb{R}$, (μ, ν) close to zero. The function $\tilde{G}$ has the form

$$\tilde{G}(\phi,\nu,\mu) = \nu\int_{-\infty}^{\infty} p_0^2 + \mu\int_{-\infty}^{\infty} p_0^4 + \varepsilon_0 \mu h_{\alpha_0,\varepsilon_0}(\phi) + O((|\nu|+|\mu|)^2)$$

as $\nu, \mu \to 0$. If we make the generic hypothesis that $h_{\alpha_0,\varepsilon_0}(\phi)$ have absolute local maximum and minimum and no ther extreme values, then there are sectors in the (ν,μ)-plane in which there are either no homoclinic orbits or there are homoclinic orbits, the same situation that occurs in Theorem 2.2.

The function $h_{\alpha_0,\varepsilon_0}(\phi)$, in principle, can be computed. One should be able to use the same argument as in [3] to show that subharmonic bifurcations also occur near points on Γ_1. Due to the complexity of the computations, we do not dwell on this question.

Equations (8.8) are the generic situation for two purely imaginary and one zero eigenvalue. If further symmetries occur in the problem, there may be no second order terms in the normal form for the vector field. In this case, the simplest case for the approximate equations are

$$(3.16)\qquad \begin{aligned}\dot{\rho} &= \rho(\lambda - a\rho^2 - by^2)\\ \dot{y} &= y(\beta + c\rho^2 + dy^2)\end{aligned}$$

with a,b,c,d fixed nonzero constants and λ,β small bifurcation parameters. In the case of a fourth order equation with two purely imaginary roots, the same equations occur in a natural way coupled with two angle variables. The complete bifurcation diagram for this equation is more difficult to obtain and will require the consideration of the fifth degree terms.

The only case that will be discussed in detail is $a = d = 1$, $bc > 1$. We will make some remarks on the other cases later. Thus, we consider the equations

$$(3.17)\qquad \begin{aligned}\dot{\rho} &= \rho(\lambda - \rho^2 - by^2)\\ \dot{y} &= y(\beta + c\rho^2 + y^2).\end{aligned}$$

The results are summarized in the following theorem.

<u>Theorem 3.2.</u> <u>There is a neighborhood</u> U <u>of</u> $(\rho,y) = (0,0)$ <u>and a neighborhood</u> V <u>of</u> $(\lambda,\beta) = (0,0)$ <u>such that the neighborhood</u> V <u>is divided into regions as shown in Fig.</u> 6 <u>such that the flow for Eq.</u> (3.17) <u>for</u> $\rho \geq 0$, $y \geq 0$, <u>in each region is the one depicted in Fig.</u> 7. <u>The curves</u> $\Gamma_1,\Gamma_2,\Gamma_3$ <u>are given by</u>

$$\Gamma_1 : \lambda + b\beta = 0, \quad \beta \leq 0$$

$$\Gamma_2 : \lambda(1+c) + \beta(1+b) = 0, \quad \beta \leq 0$$

$$\Gamma_3 : \lambda c + \beta = 0, \quad \beta \leq 0$$

All of the bifurcations are saddle-node type except the one that takes place between region 4 and region 5. On the curve Γ_2, the system is Hamiltonian and the flow is depicted in Figure 8.

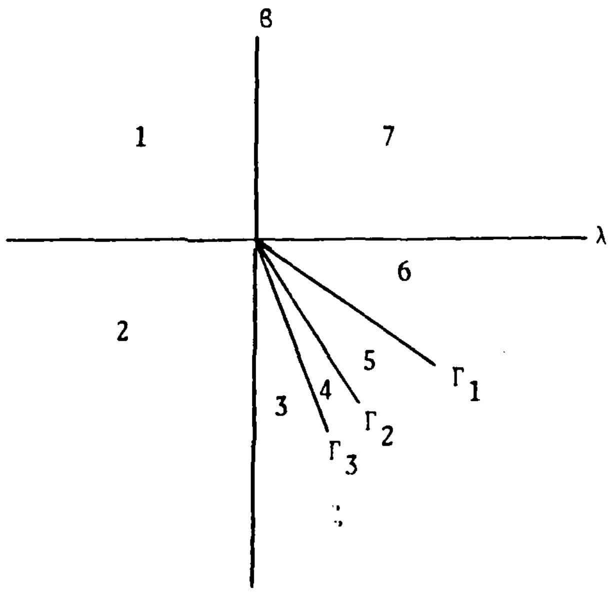

Figure 6.

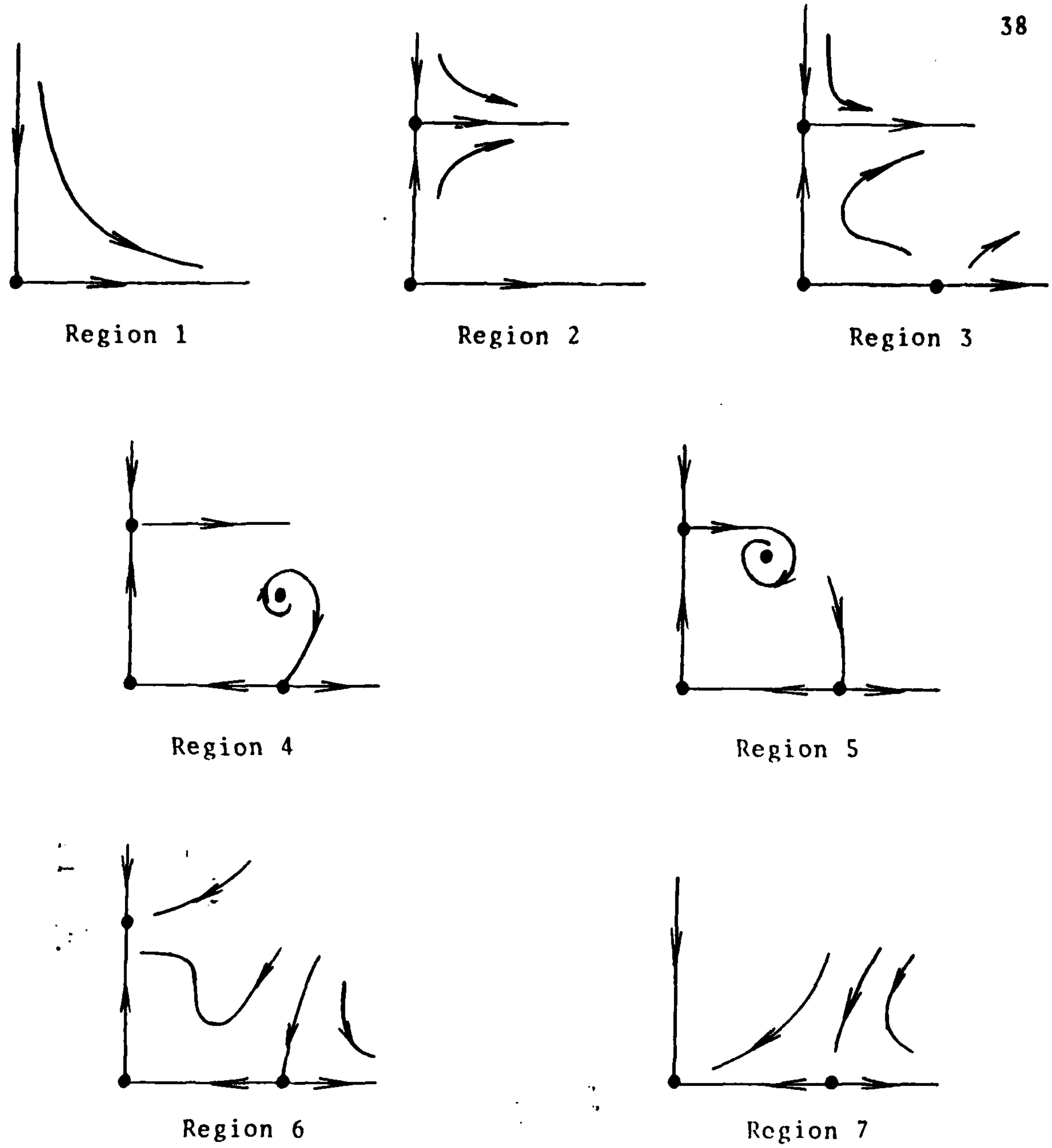

Figure 7.

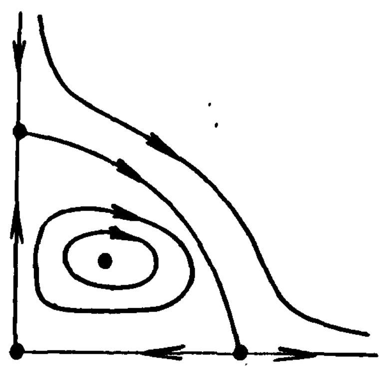

Figure 8.

Remark. The nature of the bifurcation from Region 4 to Region 5 is very complicated and not generic. To obtain a generic bifurcation, one must have some terms of order higher than three. This situation will be discussed later.

Proof: The lines $\rho = 0$ and $y = 0$ are invariant for all λ,β. In Regions 1,2,3,6,7, it is easy to verify that all equilibrium points lie on these lines. An analysis of the linear variational equation around these points gives the flows depicted in these regions. Furthermore, on the curves Γ_1,Γ_2 there is a saddle-node bifurcation which gives rise to an equilibrium point (r_0,y_0) with $\rho_0 > 0$, $y_0 > 0$. Thus, if there is to be a periodic orbit of (3.17), it must occur for values of λ,β in the union of the regions four and five and Γ_3. Also, for a periodic orbit to exist, there must be some value of α,β for which the eigenvalues of the linear variational equation about (r_0,y_0) are purely imaginary. If A is the matrix of this equation, then

$$\det A = 4\rho_0^2 y_0^2 (bc-1)$$

$$\operatorname{tr} A = -2(\rho_0^2 - y_0^2) \tag{3.18}$$

$$\rho_0^2 = -(\lambda+\beta b)/(bc-1), \quad y_0^2 = (\beta+\lambda c)/(bc-1)$$

Since $bc > 1$, $\rho_0^2 > 0$, $y_0^2 > 0$, the eigenvalues are purely imaginary if and only if $\operatorname{tr} A = 0$, that is,

$$\lambda(1+c) + \beta(1+b) = 0 \tag{3.19}$$

Equation (3.19) is the formula for Γ_2. The curves Γ_1, Γ_2 are obtained respectively when $\rho_0^2 = 0$ and $y_0^2 = 0$.

The local flow near Γ_1, Γ_2 is easily shown to be the one depicted in Regions 4 and 5. This picture remains global in both regions 4 and 5. In fact, any periodic orbit must enclose (ρ_0, y_0) since there are no other critical points. Such an orbit could only appear by the introduction of a saddle-node type bifurcation in region $\rho > 0$, $y > 0$. This is impossible.

To show there is a first integral on $\Gamma_2 = \{(\lambda,\rho): \lambda = -m_0\beta, \beta \le 0\}$, $m_0 = (1+b)/(1+c) > 0$, let $\lambda = -m_0\beta$

$$\rho \mapsto |\beta|^{1/2}\rho, \quad y \mapsto |\beta|^{1/2}y, \quad t \mapsto |\beta|^{-1}t. \tag{3.20}$$

Since $\beta \le 0$, we obtain the equivalent equations

$$\begin{aligned} \dot\rho &= m_0\rho - b\rho y^2 - \rho^3 \\ \dot y &= -y + c\rho^2 y + y^3 \end{aligned} \tag{3.21}$$

If

$$\mu = \frac{b+1}{bc-1}, \quad \nu = \frac{c+1}{bc-1} \tag{3.22}$$

then the function

(3.23) $$H(u,v) = u^{\nu}v^{\mu}[1-\frac{\nu}{\mu} u-v]$$

is a first integral of Eq. (3.21). This completes the proof of the Theorem.

Remark. If $b > 0$, $c > 0$, $bc < 1$, then the equilibrium point (ρ_0, y_0) with $\rho_0 > 0$, $y_0 > 0$ is always a node, the curve Γ_3 does not occur and the flow in the region between Γ_1 and Γ_3 is pictorially the same as shown in Regions 4 and 5 in Fig. 7 except the interior point is a stable hyperbolic node. If $b > 0$, $c < 0$, then additional complications arise because there is the possibility of two more equilibrium points in the region $\rho > 0$, $y > 0$ and there can be a saddle-node bifurcation in this region. We do not discuss this case in detail.

Let us now discuss perturbations of Eq. (3.17), beginning with autonomous perturbations. All of the phase portraits in Fig. 7 will be preserved under small perturbations. However, the one in Fig. 8 will not. Therefore to obtain a more complete picture of the behavior of the equation under perturbation, we consider the effect of the addition of higher order terms in the same way as cubic terms were considered in Theorem 3.1.

If we consider Eq. (3.16) as arising from the problem in $\mathbb{R}^4$ where the linear variational equation has two pair of purely imaginary roots, then the perturbation terms must be 5th degree. Thus, we consider

$$
(3.24)\qquad \begin{aligned} \dot{\rho} &= \rho(\lambda-\rho^2-by^2+\rho^4) \\ \dot{y} &= y(\beta+c\rho^2+y^2). \end{aligned}
$$

We are interested in the behavior of solutions near the line $\lambda = -m_0\beta$, $m_0 = (b+1)/(1+c)$. Introduce the scaling

$$
\rho \mapsto |\beta|^{1/2}\rho, \quad y \mapsto |\beta|^{1/2}y, \quad t \mapsto |\beta|^{-1}t, \; \lambda \mapsto -m_0\beta + \alpha\beta
$$

to obtain

$$
(3.25)\qquad \begin{aligned} \dot{\rho} &= \rho(m_0+\alpha-\rho^2-by^2+\beta\rho^4) \\ \dot{y} &= y(-1+c\rho^2+y^2). \end{aligned}
$$

One can proceed as in the proof of Theorem 3.1 to obtain the curve in the (λ,β)-plane along which Eq. (3.24) has a homoclinic orbit. In terms of the scaled variables, we observe that such a solution can exist only in a neighborhood of $\alpha = 0$, $\beta = 0$. We then obtain a bifurcation function $G(\alpha,\beta)$ for a homoclinic orbit and observe that it has the form

$$G(\alpha,\beta) = \alpha\int_{-\infty}^{\infty}\rho_0^2(t)dt + \beta\int_{-\infty}^{\infty}\rho_0^6(t)dt + O((|\alpha|+|\beta|)^2)$$

as $\alpha,\beta \to 0$, where $(\rho_0(t),y_0(t))$ describe the homoclinic orbit for $\alpha = 0$, $\beta = 0$. There is no term involving $\rho_0^4(t)$ because for $\alpha = 0$, $\beta = 0$, the system is Hamiltonian. The equation $G(\alpha,\beta) = 0$ has a unique solution $\alpha = \alpha^*(\beta)$ in a neighborhood of $(\alpha,\beta) = (0,0)$, $\alpha^*(0) = 0$. This gives a curve

$$\Gamma_2' = \{(\lambda,\beta) : \lambda = -m_0\beta + \alpha^*(\beta)\beta\}$$

in the (λ,β)-plane such that Eq. (3.4) for $(\lambda,\beta) \in \Gamma_2'$ has a homoclinic orbit.

One can also obtain a curve Γ_2'' in the (λ,β)-plane

$$\Gamma_2'' = \{(\lambda,\beta) : \lambda = -m_0\beta + \alpha^{**}(\beta)\beta\}$$

where $\alpha^{**}(0) = 0$, such that for any $(\lambda,\beta) \in \Gamma_2''$, the linear variational equation for the solution (ρ_0,y_0), $\rho_0 > 0$, $y_0 > 0$, of (3.4) has both eigenvalues purely imaginary. Proceeding as before, one obtains the existence of a unique periodic orbit in the Region 4' between Γ_2' and Γ_2''. The flow in Region 4' is shown in Fig. 9. The complete bifurcation diagram is shown in Fig. 10.

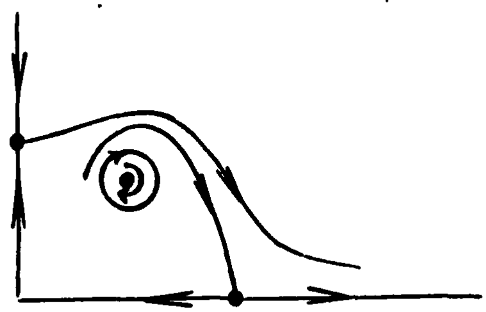

Figure 9.

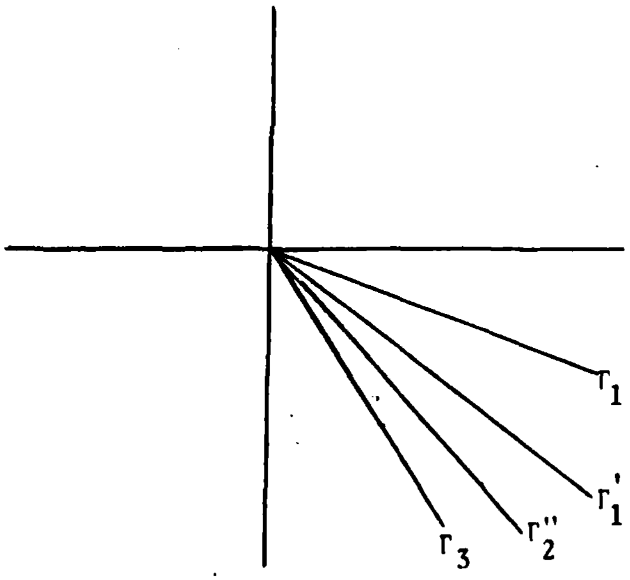

Figure 10.

It remains to discuss perturbations of (3.24) when the higher order terms contain angle variables. If there is only one angle variable (the case of two purely imaginary and one zero root) everything follows exactly as for Eq. (3.14). If there are two angle variables, the situation is more complicated. The general theory of integral manifolds implies that every hyperbolic equilibrium point of the unperturbed equation becomes a hyperbolic torus of dimension 2 and every hyperbolic periodic orbit of the unperturbed equation becomes a hyperbolic three dimensional torus as one crosses Γ_2'' (the Hopf bifurcation). The analysis in Section 2 shows that the two dimensional torus bifurcates to give a three dimensional torus. None of the other cases are well understood.

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